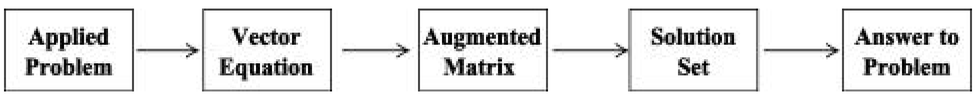
1.10 LINEAR MODELS IN BUSINESS, SCIENCE, AND ENGINEERING

This is the second of twelve sections devoted to uses of linear algebra. The applications in the text were selected to give you an impression of the power of linear algebra. You are likely to encounter some of these topics again—in school or in your career—and the discussions in your text will be valuable references.

The main point of this section is to present several interesting applications in which "linearity" arises naturally.

STUDY NOTES

Nutrition Problem: In some applied problems such as the nutrition problem considered here, the data are already organized naturally in a manner that leads to a vector equation of the type we have discussed. The steps to the solution in this case may be diagrammed as follows:



The nutrition model is linear because the nutrients supplied by each foodstuff are *proportional* to the amount of the foodstuff added to the diet mixture, and each nutrient in the mixture is the *sum* of the amounts from each foodstuff. Study equations (1) and (2) on page 94.

The nutrition problem leads naturally into linear programming, a subject that uses linear algebra and has applications in agriculture, business, engineering, and other areas. In the 1950's and 1960's, one of the most common applications of linear algebra (measured in millions of dollars per year for computer time) was to linear programming problems. Such problems are still of great importance in operations research and management science. The following reference gives an entertaining introduction to linear programming. Matrix notation is used in the appendix (pp. 127-152).

Gass, Saul I., *An Illustrated Guide to Linear Programming,* New York: McGraw-Hill, 1970. Republished by Dover Publications, 1990.

Electrical Networks: The linearity of this model, which is evident from the matrix equation $R\mathbf{i} = \mathbf{v}$, comes from the linearity of Ohm's law and Kirchhoff's voltage law. (Kirchhoff's current law, which is also linear, is needed when studying another model that involves branch currents.)

Population Movement: The entries in each *column* of the migration matrix must sum to one because the (decimal) fractions in a column account for the entire population in one region. A certain fraction of the population in a region remains in (or moves within) that region, and other fractions move elsewhere.

SOLUTIONS TO EXERCISES

1. a. If x_1 is the number of servings of Cheerios and x_2 is the number of servings of 100% Natural Cereal, then x_1 and x_2 should satisfy

$$x_1\begin{bmatrix}\text{nutrients}\\ \text{per serving}\\ \text{of Cheerios}\end{bmatrix} + x_2\begin{bmatrix}\text{nutrients}\\ \text{per serving of}\\ \text{100\% Natural}\end{bmatrix} = \begin{bmatrix}\text{quantities}\\ \text{of nutrients}\\ \text{required}\end{bmatrix}$$

That is,

$$x_1\begin{bmatrix}110\\4\\20\\2\end{bmatrix} + x_2\begin{bmatrix}130\\3\\18\\5\end{bmatrix} = \begin{bmatrix}295\\9\\48\\8\end{bmatrix}$$

b. The equivalent matrix equation is $\begin{bmatrix}110 & 130\\ 4 & 3\\ 20 & 18\\ 2 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}295\\9\\48\\8\end{bmatrix}$. To solve this, row reduce the augmented matrix for this equation.

$$\begin{bmatrix}110 & 130 & 295\\ 4 & 3 & 9\\ 20 & 18 & 48\\ 2 & 5 & 8\end{bmatrix} \sim \begin{bmatrix}2 & 5 & 8\\ 4 & 3 & 9\\ 20 & 18 & 48\\ 110 & 130 & 295\end{bmatrix} \sim \begin{bmatrix}1 & 2.5 & 4\\ 4 & 3 & 9\\ 10 & 9 & 24\\ 110 & 130 & 295\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 2.5 & 4\\ 0 & -7 & -7\\ 0 & -16 & -16\\ 0 & -145 & -145\end{bmatrix} \sim \begin{bmatrix}1 & 2.5 & 4\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1.5\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The desired nutrients are provided by 1.5 servings of Cheerios together with 1 serving of 100% Natural Cereal.

Study Tip: Be sure to distinguish between (i) the vector equation, (ii) the matrix equation (which has the form $A\mathbf{x} = \mathbf{b}$), and (iii) the augmented matrix (which has the form $[A \quad \mathbf{b}]$) that represents a system of linear equations.

7. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix}12\\ -7\\ 0\\ -4\end{bmatrix} \begin{matrix}\text{Total of three RI voltage drops for current } I_1\\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction}\\ \text{Current } I_3 \text{ does not flow in loop 1}\\ \text{Voltage drop for } I_4 \text{ is negative; } I_4 \text{ flows in opposite direction}\end{matrix}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix}-7\\ 15\\ -6\\ 0\end{bmatrix} \begin{matrix}\text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction}\\ \text{Total of three RI voltage drops for current } I_2\\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction}\\ \text{Current } I_4 \text{ does not flow in loop 2}\end{matrix}$$

Also, $\mathbf{r}_3 = \begin{bmatrix}0\\ -6\\ 14\\ -5\end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix}-4\\ 0\\ -5\\ 13\end{bmatrix}$, and $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{r}_4] = \begin{bmatrix}12 & -7 & 0 & -4\\ -7 & 15 & -6 & 0\\ 0 & -6 & 14 & -5\\ -4 & 0 & -5 & 13\end{bmatrix}$.

Note that each off-diagonal entry of R is negative (or zero). This happens because the loop current directions are all chosen in the same direction on the figure. (For each loop j, this choice forces the currents in other loops adjacent to loop j to flow in the direction opposite to current I_j.)

Next, set $\mathbf{v} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$. Note the negative voltage in loop 4. The current direction chosen in loop 4 is opposed by the orientation of the voltage source in that loop. Thus $R\mathbf{i} = \mathbf{v}$ becomes

$$\begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}.$$ **[M]**: The solution is $\mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 11.43 \\ 10.55 \\ 8.04 \\ 5.84 \end{bmatrix}$.

13. **[M]** The order of entries in a column of a migration matrix must match the order of the columns. For instance, if the first column concerns the population in the city, then the first entry in *each* column of the matrix must be the fraction of the population that moves to (or remains in) the city. In this case, the data in the exercise leads to

$$M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \text{ and } \mathbf{x}_0 = \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}.$$

a. Some of the population vectors are

$$\mathbf{x}_5 = \begin{bmatrix} 523{,}293 \\ 476{,}707 \end{bmatrix}, \ \mathbf{x}_{10} = \begin{bmatrix} 472{,}737 \\ 527{,}263 \end{bmatrix}, \ \mathbf{x}_{15} = \begin{bmatrix} 439{,}417 \\ 560{,}583 \end{bmatrix}, \ \mathbf{x}_{20} = \begin{bmatrix} 417{,}456 \\ 582{,}544 \end{bmatrix}$$

The data here shows that the city population is declining and the suburban population is increasing, but the changes in population each year seem to grow smaller.

b. When $\mathbf{x}_0 = \begin{bmatrix} 350{,}000 \\ 650{,}000 \end{bmatrix}$, the situation is different. Now

$$\mathbf{x}_5 = \begin{bmatrix} 358{,}523 \\ 641{,}477 \end{bmatrix}, \ \mathbf{x}_{10} = \begin{bmatrix} 364{,}140 \\ 635{,}860 \end{bmatrix}, \ \mathbf{x}_{15} = \begin{bmatrix} 367{,}843 \\ 632{,}157 \end{bmatrix}, \ \mathbf{x}_{20} = \begin{bmatrix} 370{,}283 \\ 629{,}717 \end{bmatrix}$$

The city population is increasing slowly and the suburban population is decreasing. No other conclusions are expected. (This example will be analyzed in greater detail later in the text.)

MATLAB Generating a Sequence

The m-file (in the Toolbox) for Exercises 9–13 in Section 1.9 stores initial vectors in **`x0`**. Set **`x = x0`** to put the initial data into **`x`**. Then use the command **`x = M*x`** repeatedly to generate the sequence **`xl, x2, ....`** You only type the command once. After that, use the up-arrow (↑) key to recall the command, and press <Enter>.

In Exercise 11, you need 6 decimal places to get four significant figures in M(1, 2). Use the command **`format long`** and then **`M`** to see more decimal places in *M*. The command **`format short`** wil1 return MATLAB to the standard four decimal place display. (The display format does not affect MATLAB's accuracy in computations.)

Numbers are entered in MATLAB without commas. The number 600,000 in MATLAB scientific notation is 6e5. A small number such as .00000012 is 1.2e-7.