1.6 APPLICATIONS OF LINEAR SYSTEMS

All of the examples and exercises in this system involve linear systems that have multiple solutions. In each case, make a note of *why* you should expect the system to have many solutions.

STUDY NOTES

The Leontief exchange model concerns the dollar value (called the *price*) of the annual output of each sector of a nation's economy. An equilibrium price vector **p** provides a list of prices, one for each section, such that each sector's expenses and income are in balance. Example 1 shows that there are many equilibrium price vectors; each one is a multiple of a fixed equilibrium price vector. This means that once the prices are all in balance, multiplying all the prices by a fixed constant does not affect the balance. For instance, if all prices are doubled, then each sector's expenses and income are doubled at the same time and hence they remain in balance.

A solution of a chemical equation balance problem is a list of coefficients that appear on the various terms in the chemical equation. When a chemical equation is balanced, the number of atoms of each type on the left side of the equation matches the number of corresponding atoms on the right side. If the coefficients in the equation are each multiplied by a fixed positive integer, the equation will remain balanced. So, there are many solutions to a chemical equation balance problem.

The problems here in network flow have multiple solutions for the simple reason that there are more variables than there are constraint equations. The equations for network flow are mostly nonhomogeneous. In contrast, the Leontief model and the chemical equation-balance problem both lead to systems of homogeneous equations.

SOLUTIONS TO EXERCISES

1. Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

Distribution of Output From:				
	Goods	Services		Purchased by:
output	↓	↓	input	
	.2	.7	→	Goods
	.8	.3	→	Services

Denote the total annual output (in dollars) of the sectors by p_G and p_S. From the first row, the total input to the Goods sector is $.2\,p_G + .7\,p_S$. The Goods sector must pay for that. So the equilibrium prices must satisfy

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_G & = & .2p_G + .7\,p_S \end{array}$$

From the second row, the input (that is, the expense) of the Services sector is $.8\,p_G + .3\,p_S$. The equilibrium equation for the Services sector is

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_S & = & .8p_G + .3\,p_S \end{array}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .8p_G - .7\,p_S &= 0 \\ -.8p_G + .7\,p_S &= 0 \end{aligned}$$

Row reduce the augmented matrix:

$$\begin{bmatrix} .8 & -.7 & 0 \\ -.8 & .7 & 0 \end{bmatrix} \sim \begin{bmatrix} .8 & -.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.875 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_G = .875\,p_S$, with p_S free. One equilibrium solution is $p_S = 1000$ and $p_G = 875$. If one uses fractions instead of decimals in the calculations, the general solution would be written $p_G = (7/8)\,p_S$, and a natural choice of prices might be $p_S = 80$ and $p_G = 70$. Only the *ratio* of the prices is important: $p_G = .875\,p_S$. The economic equilibrium is unaffected by a proportional change in prices.

7. The following vectors list the numbers of atoms of sodium (Na), hydrogen (H), carbon (C), and oxygen (O):

$$NaHCO_3: \begin{bmatrix}1\\1\\1\\3\end{bmatrix},\ H_3C_6H_5O_7: \begin{bmatrix}0\\8\\6\\7\end{bmatrix},\ Na_3C_6H_5O_7: \begin{bmatrix}3\\5\\6\\7\end{bmatrix},\ H_2O: \begin{bmatrix}0\\2\\0\\1\end{bmatrix},\ CO_2: \begin{bmatrix}0\\0\\1\\2\end{bmatrix} \begin{matrix}\text{sodium}\\\text{hydrogen}\\\text{carbon}\\\text{oxygen}\end{matrix}$$

The order of the various atoms is not important. The list here was selected by writing the elements in the order in which they first appear in the chemical equation, reading left to right:

$$x_1 \cdot NaHCO_3 + x_2 \cdot H_3C_6H_5O_7 \longrightarrow x_3 \cdot Na_3C_6H_5O_7 + x_4 \cdot H_2O + x_5 \cdot CO_2$$

The coefficients $x_1, \ldots, x_5$ satisfy the vector equation

$$x_1\begin{bmatrix}1\\1\\1\\3\end{bmatrix} + x_2\begin{bmatrix}0\\8\\6\\7\end{bmatrix} = x_3\begin{bmatrix}3\\5\\6\\7\end{bmatrix} + x_4\begin{bmatrix}0\\2\\0\\1\end{bmatrix} + x_5\begin{bmatrix}0\\0\\1\\2\end{bmatrix}$$

Move all terms to the left side (changing the sign of each entry in the third, fourth, and fifth vectors) and reduce the augmented matrix:

$$\begin{bmatrix}1 & 0 & -3 & 0 & 0 & 0\\1 & 8 & -5 & -2 & 0 & 0\\1 & 6 & -6 & 0 & -1 & 0\\3 & 7 & -7 & -1 & -2 & 0\end{bmatrix} \sim \cdots \sim \begin{bmatrix}1 & 0 & 0 & 0 & -1 & 0\\0 & 1 & 0 & 0 & -1/3 & 0\\0 & 0 & 1 & 0 & -1/3 & 0\\0 & 0 & 0 & 1 & -1 & 0\end{bmatrix}$$

The general solution is $x_1 = x_5$, $x_2 = (1/3)x_5$, $x_3 = (1/3)x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 3$. Then $x_1 = x_4 = 3$, and $x_2 = x_3 = 1$. The balanced equation is

$$3NaHCO_3 + H_3C_6H_5O_7 \longrightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

13. Write the equations for each intersection (see the diagram for the intersection labels):

Intersection	Flow in		Flow out	Rearrange the equations:
A	$x_2 + 30$	=	$x_1 + 80$	$x_1 - x_2 = -50$
B	$x_3 + x_5$	=	$x_2 + x_4$	$x_2 - x_3 + x_4 - x_5 = 0$
C	$x_6 + 100$	=	$x_5 + 40$	$x_5 - x_6 = 60$
D	$x_4 + 40$	=	$x_6 + 90$	$x_4 - x_6 = 50$
E	$x_1 + 60$	=	$x_3 + 20$	$x_1 - x_3 \quad -40$
Total flow:	230	=	230	

Completely reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$

b. To find minimum flows, note that since x_1 cannot be negative, $x_3 \geq 40$. This implies that $x_2 \geq 50$. Also, since x_6 cannot be negative, $x_4 \geq 50$ and $x_5 \geq 60$. The minimum flows are $x_2 = 50$, $x_3 = 40$, $x_4 = 50$, $x_5 = 60$ (when $x_1 = 0$ and $x_6 = 0$).

MATLAB Rational Format

Chemical equation-balance problems are studied best using exact or symbolic arithmetic, because the balance variables must be whole numbers (with no round-off allowed). In MATLAB, a simple approach is to execute the command **`format rat`**, which will make MATLAB display matrix or vector entries as rational numbers. In general, the rational number displayed might be only an approximation for a floating-point number. But since the chemical equations studied here have integer coefficients, **`format rat`** will make MATLAB display the exact (rational) value of every entry during row reduction. Use **`format`** or **`format short`** to return to the standard MATLAB display of numbers.

Once you find a rational solution of a chemical equation-balance problem, you can multiply the entries in the solution vector by a suitable integer to produce a solution that involves only whole numbers.